Components of totally symmetric and anti-symmetric tensors

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We show how to find the number of independent components of a tensor that is totally symmetric in all of its indices. We give some simple examples but the important result is the general formula and its proof using the bars and stars trick. We also discuss totally anti-symmetric tensors.

1 Simple examples

Let's consider a tensor living in d dimensions, meaning that each index runs from 1 to d. The rank of the tensor r is the number of indices that it has and the fact that it is totally symmetric means that $T_{\dots a \dots b \dots} = T_{\dots b \dots a \dots}$ for any pair of indices.

The first example to look at is a tensor with two indices T_{ab}. This case is simple because it represents the components of a symmetric $d \times d$ matrix. The only independent components are the diagonal elements and the upper triangle because the lower triangle is determined from the upper one by the symmetry. There are of course d diagonal elements and we are left with $d^2 - d$ non-diagonal elements, which leads to $\frac{d(d-1)}{2}$ elements in the upper triangle. The total of independent components is then

$$d + \frac{d(d-1)}{2} = \boxed{\frac{d(d+1)}{2}}$$

The next example will be more complicated and will show the general idea. Consider a totally symmetric tensor of rank 3, T_{abc}. There are three types of components possible. We can first have things like T_{aaa} for a fixed number a from 1 to d and of course there are d such terms. We can next have terms like T_{aab} with $a \neq b$ and we don't care about T_{aba} or T_{baa} because they are related to T_{aab} by symmetry. The number of these components is $d(d-1)$ because you have d options for the first (repeated) index and then only $d-1$ possible choices for the remaining index. The last type of components is T_{abc} with $a \neq b \neq c$. There are $\binom{d}{3} = \frac{d(d-1)(d-2)}{3!}$ independent components of that form because we only need to choose three different numbers between 1 and d and the order doesn't matter. The total number of independent components is then

$$d + d(d-1) + \frac{d(d-1)(d-2)}{3!} = \boxed{\frac{d(d+1)(d+2)}{3!}}$$

2 General formula

Now, instead of continuing with every possible example, let's discuss a more clever way of organizing the different types of components. Since the order of the indices doesn't matter, this problem is basically the same as asking how many ways there are to put r undistinguishable objects (the indices) into d boxes (the possible values for the indices). In other words we specify each independent component of the tensor by saying how many indices have a given value, keeping in mind that it's possible that no index has some value.

To solve this problem, picture the objects as stars and divide them into boxes using bars. There are d bars and the number of stars between two bars determines the number of objects in a box. There can be bars next to each other, which means that the box is empty. The actual division of the objects is then determined by looking from left to right and uing this rule to distribute the objects into the boxes 1 to d (for example, the number of objects in box 1 is determined by the number of stars to the left of the first bar). Each configuration of stars and bars like the one in Figure 1 represents a different combination. To count how many there are, note that all we have to do is place $d + r - 1$ symbols (the -1 is there because all the objects have to be in some box so we must always end the sequence by a bar) and r of these symbols must be bars. Combinatorics then tells us that there are $\binom{d+r-1}{r}$ different ways of doing that.

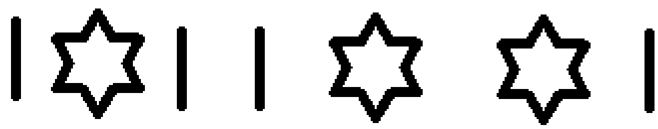

Figure 1: An example of configuration of stars and bars with no objects in box 1, 1 object in box 2, no object in box 3 and 2 objects in box 4 for a total of 3 objects divided into 4 boxes. This is equivalent to the independent components of a tensors with 3 indices that can run from 1 to 4.

Coming back to tensors, the analogy that we used gave us the result that the number of independent components of a totally symmetric tensor of rank r in d dimensions is

$$\binom{d+r-1}{r} = \frac{(d+r-1)!}{r!(d-1)!} = \boxed{\frac{(d+r-1)(d+r-2)\cdots(d+1)d}{r!}}$$

Of course this agrees with the examples discussed previously.

It is useful to add the constraint of vanishing trace to the symmetric tensors and know how many components there are left. We want to contract a pair of indices and substract the resulting components from the tensor. Since the tensor is symmetric, any contraction is the same so we only get constraints from one contraction. The result of the contraction is a tensor of rank $r - 2$ so we get as many components to substract as there are components in a tensor of rank $r - 2$. The total number of independent components in a totally symmetric traceless tensor is then

$$\binom{d+r-1}{r} - \binom{d+r-3}{r-2}$$

3 Totally anti-symmetric tensors

It's possible to do the same kind of thing for totally anti-symmetric tensors that satisfy $T_{\ldots a \ldots b \ldots} = -T_{\ldots b \ldots a \ldots}$ for every pair of indices, but the analysis is easier. From the anti-symmetry we can already deduce that the value of all the indices for a non-zero component must be different because otherwise we would have $T_{\ldots a \ldots a \ldots} = -T_{\ldots a \ldots a \ldots} \Rightarrow T_{\ldots a \ldots a \ldots} = 0$. This then means that we can't have a non-trivial totally anti-symmetric tensor with $r > d$. For a generic $r \leq d$, since we can relate all the componnts that have the same set of values for the indices together by using the anti-symmetry, we only care about which numbers appear in the component and not the order. The number of independent components is then simple the number of ways of picking r numbers out of d without a specific order, which is

$$\binom{d}{r} = \boxed{\frac{d!}{r!(d-r)!}}$$

In particular there is only one free component for an anti-symmetric tensor of rank d in d dimensions. Asking for $T_{12\ldots d} = 1$ defines the well-known Levi-Civita tensor $\epsilon_{a_1 a_2 \ldots a_d}$.